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Approximations from Subspaces of $C_0(X)$

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We show among other things that if B is a linear space of continuous real-valued functions vanishing at infinity on a locally compact Hausdorff space X, for which there is a continuous function h defined in a neighbourhood of 0 in the real line which is non-affine in every neighbourhood of 0 and satisfies $|h(t)| \leqslant k\,|t|$ for all t, such that $h \circ b$ is in B whenever b is in B and the composite function is defined, then every function in $C_0(X)$ which can be approximated on every pair of points in X by functions in B can be approximated uniformly by functions in B.

INTRODUCTION

The real version of the Stone–Weierstrass Theorem states that if B is a subspace of $C(X)$, the algebra of all real-valued continuous functions on a compact Hausdorff space X, and if B separates the points of X, contains the constant functions and is closed under multiplication, then every function in $C(X)$ can be approximated uniformly by functions in B.

For a subspace B, saying that B is closed under multiplication is equivalent to saying that b^2 is in B for every b in B. Let us say that a function h defined on some interval I of the real line *operates* on B (more precisely, operates by composition on B) if $h \circ b$ is in B whenever b is in B and the composite function is defined, that is, whenever b maps X into I. In general it might well be that the only functions operating on a subspace B are the affine functions, $h(t) = \alpha t + \beta$ or just $h(t) = \alpha t$, in case B does not contain the constant functions.

The Stone–Weierstrass Theorem can be formulated in terms of operating functions: If B is a subspace of $C(X)$ separating the points of X and containing the constant functions and if the function $h(t) = t^2$ operates on B, then every function in $C(X)$ can be approximated uniformly by functions from B.

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K. de Leeuw and Y. Katznelson [2] extended this version of the Stone–Weierstrass Theorem in that they showed that in the assumptions for the theorem one could replace the function $h(t)=t^2$ by any continuous non-affine function defined on some interval of the real line, and still have the same conclusion.

The Stone–Weierstrass Theorem also holds in the case where X is a locally compact Hausdorff space: If B is a subspace of $C_0(X)$, the algebra of all real-valued continuous functions on X vanishing at infinity, and if B separates the points of X and does not vanish identically at any point in X then, if B is closed under multiplication, any function in $C_0(X)$ can be approximated uniformly by functions from B.

The following simple example shows that the theorem of de Leeuw and Katznelson cannot immediately be transferred to the locally compact case.

EXAMPLE 1. Let $X=[-1,1]\setminus\{0\}$, so that $C_0(X)$ can be identified with the space of all continuous functions on the interval $[-1,1]$ which vanish at 0.

(i) If

$$B=\{f\in C_0(X)\mid f(x)=\alpha x \text{ for } x<0 \text{ and } f(x)=\beta x \text{ for } x>0\},$$

where α, β are real numbers, the function $h(t)=|t|$ operates on B.

(ii) If

$$B=\{f\in C_0(X)\mid f(-x)=-f(x) \text{ for all } x\in X\},$$

the function $h(t)=t^3$ operates on B.

(iii) If

$$B=\{f\in C_0(X)\mid f(\lambda x)=\lambda f(x) \text{ for all } x\in X\},$$

where λ is a fixed number in the interval $(-1,1)$, any continuous function h on the interval $[-1,1]$, satisfying $h(\lambda t)=\lambda h(t)$ for all t in $[-1,1]$, operates on B.

THE MAIN PART

We will try to describe those subspaces of $C_0(X)$ which have non-affine operating functions and show that the examples above are typical for such spaces. Although we do not have descriptions for the most general operating functions, we shall treat some fairly general cases, extending previous results obtained by the author [1].

As in the case where X is compact we say that a subspace B of $C_0(X)$ separates the points of X if there is for every pair x, y of points in X a function b in B such that $b(x) \neq b(y)$. It will always be assumed that the spaces B considered separate the point at infinity from other points of X, meaning that there is no point x in X where all the functions in B vanish. This gives two kinds of point separation. Either the vectors $(b(x), b(y))$, where b runs through B, span a two-dimensional space, in which case there is a pair b_x, b_y in B such that $b_x(x) = b_y(y) = 1$ and $b_x(y) = b_y(x) = 0$ or, the vectors are all proportional, in which case there exists a number λ such that $b(x) = \lambda b(y)$ for every b in B. In the first case we say that B separates x, y *strongly*, in the second case that B separates x, y *weakly*.

We begin with the simple case where the function $h(t) = |t|$ operates on B. Since

$$a \vee b = (1/2)(a+b+|a-b|) \qquad \text{and} \qquad a \wedge b = (1/2)(a+b-|a-b|)$$

the space B must be a lattice.

THEOREM 1. *Let B be a uniformly closed subspace of $C_0(X)$. If $h(t) = |t|$ operates on B, every function in $C_0(X)$ which can be approximated on every pair of points in X by a function in B is actually in B*

This result for the compact case is one of the tools in the usual proof of the Stone–Weierstrass Theorem. The proof Theorem 1 is the same as f. ex. the proof of Lemma 4.3.2 in [4] with X replaced by the one-point compactification of X.

For the next theorems we need an approximation result due to Machado [3] adapted to the locally compact case. A short and elementary proof which is easily adapted to the locally compact case can be found in [5].

THEOREM 2. *Let X be a locally compact space and B a subspace of $C_0(X)$. Let further A be an algebra of continuous bounded functions on X, containing the constant functions, with the property that ab is in B for all a in A and b in B. Then for every f in $C_0(X)$ there is a closed subset S of X, on which every function in A is constant, such that*

$$\inf_{b \in B} \sup_{x \in X} |f(x) - b(x)| = \inf_{b \in B} \sup_{x \in S} |f(x) - b(x)|.$$

Machado's theorem is useful because in some situations sets of constancy for A are small and easy to determine as is the case in the next theorem.

THEOREM 3. *Let B be a uniformly closed subspace of $C_0(X)$ which separates the points of X and let $n > 1$ be a natural number such that b^n is in*

B for every b in B. If n is even, $B = C_0(X)$, while if n is odd, B contains every function f in $C_0(X)$ with the property that $f(x_1) = -f(x_2)$ whenever $b(x_1) = -b(x_2)$ for all b in B.

Proof. For a, b in B and a non-zero real number t, the function $((a+tb)^n - a^n)$ is in B. Multiplying by $1/t$ and then letting t tend to 0 we find that $a^{n-1}b$ is in B for every a and b in B. Let A be the algebra of all bounded continuous functions f on X for which fb is in B for all b in B. The algebra A contains a^{n-1} for every a in B. We deduce that the sets of constancy for A are one-point sets if $n-1$ is odd and one-point sets or sets of the form $\{x_1, x_2\}$, where $b(x_1) = -b(x_2)$ for all b in B, if $n-1$ is even. The result now follows from Machado's theorem.

Suppose a function h, differentiable at 0, and not of the form $h(t) = \alpha t$ in any neighbourhood of 0, operates on a subspace B of $C_0(X)$. Then B separates x_1, x_2 in X weakly only if $b(x_1) = -b(x_2)$ for all b in B. Because if $b(x_1) = \lambda b(x_2)$ for all b in B where $|\lambda| < 1$, it follows that $h(tb(x_1)) = \lambda h(tb(x_2))$ for all t in a neighbourhood of 0 and hence $h(\lambda s) = \lambda h(s)$ for all s in some neighbourhood of 0. But then $h(\lambda^n s) = \lambda^n h(s)$ for all s in the same neighbourhood of 0. Since h is differentiable at 0 we find that $h(s) = h'(0)\, s$ for all s in a neighbourhood of 0, contrary to the assumptions on h. We have thus proved.

PROPOSITION 1. *If there is a function, non-affine in every neighbourhood of 0 and differentiable at 0, operating on a subspace B of $C_0(X)$, every two points x_1, x_2 in X which are separated by B are either strongly separated by B or, $b(x_1) = -b(x_2)$ for all b in B.*

We are now able to extend Theorem 3 to differentiable operating functions.

THEOREM 4. *Let B be a uniformly closed subspace of $C_0(X)$ separating the points of X and suppose there is a continuously differentiable operating function for B which is non-affine in every neighbourhood of 0. Then B contains every function f in $C_0(X)$ with the property that $f(x_1) = -f(x_2)$ whenever $b(x_1) = -b(x_2)$ for every b in B.*

Proof. Let a, b be in B and let s, t be real numbers with $|s|, |t|$ sufficiently small so that $h \circ (sa + tb)$ is defined. Differentiating $t \to h \circ (sa+tb)$ w.r.t. t and then setting $t = 0$, we find that $b \cdot (h' \circ (sa))$ is in B. It follows that the algebra A of all bounded continuous functions f on X for which fb is in B for all b in B contains the functions $h' \circ (sa)$. Let x_1, x_2 be in X. If $a(x_1) = 0$ and $a(x_2) = 1$ we can choose s so that $h'(sa(x_2)) \neq h'(0) = h'(sa(x_1))$ because otherwise $h(t) = \alpha t$ for t in a neighbourhood of 0. Thus A separates the points x_1, x_2 if they are strongly separated by B. It follows that every pair of points in X which belong to a set of constancy for A must be weakly separated by B. By the proposition above this means that if x_1, x_2 belong to a set of constancy for A, $b(x_1) = -b(x_2)$ for all b in B. Since B separates the points of X, this implies that the sets of constancy for A contain one or two points and in case of two points x_1, x_2, $b(x_1) = -b(x_2)$ for all b in B. We now use Machado's theorem and the result follows.

Before proceeding any further let us examine how weak separation of points affects operating functions.

Let us assume that B separates a pair x_1, x_2 of points in X weakly say, $b(x_1) = \lambda b(x_2)$ for all b in B and some λ in the interval $[-1, 1)$. If h is an operating function for B then as we saw above

$$h(\lambda s) = \lambda h(s)$$

for all s in some neighbourhood of 0.

If $\lambda = -1$, the function h is an odd function in a neighbourhood of 0. If λ belongs to the interval $(-1, 1)$ we say that h is *λ-affine* in a neighbourhood of 0. The affine functions are λ-affine for every λ.

We gather some informations about λ-affine functions in a proposition. If h is λ-affine, the function $t \to h(rt)$ is λ-affine on the interval $[-1, 1]$ for a suitable choice of the real number r. For this reason we confine ourselves to functions which are λ-affine on the interval $[-1, 1]$ in the proposition.

PROPOSITION 2. (i) *If $h(t) = rt$ for $t < 0$ and $h(t) = st$ for $t > 0$ then h is λ-affine for every $\lambda \in (0, 1)$.*

(ii) *Suppose h is not as in* (i). *If h is λ-affine then there is a largest number $\gamma \in (0, 1)$ for which h is γ-affine. For every positive ρ for which h is ρ-affine we have $\rho = \gamma^n$ for some natural number n. If there is a negative number λ for which h is λ-affine then there is a number $\mu \in (-1, 0)$ such that h is μ-affine and such that if h is ρ-affine and ρ is negative then $\rho = \gamma^k \mu$, for some natural number k.*

(iii) *If h is λ-affine then $|h(t)| \leqslant k\,|t|$ for all $t \in [-1, 1]$ and some $k > 0$.*

(iv) *Let $\lambda \in (-1, 1)$ and let ψ be a continuous function on $[-1, -|\lambda|] \cup [|\lambda|, 1]$ satisfying $\psi(\lambda) = \lambda\psi(1)$ and $\psi(-\lambda) = \lambda\psi(-1)$. Then ψ can be extended uniquely to a continuous λ-affine function on $[-1, 1]$. If ψ is differentiable on $[-1, -|\lambda|] \cup [|\lambda|, 1]$ having right and left derivatives at the respective endpoints, all equal to 0, then the extension is differentiable on $(-1, 1)$, except possibly at 0.*

Proof. (i) is clear. To prove (ii) suppose h is not as in (i) and that h is α-affine and β-affine for positive α and β. Let m, n be natural numbers with $\alpha^m\beta^{-n}<1$. Then

$$\alpha^m h(t)=h(\alpha^m t)=h(\beta^n\alpha^m\beta^{-n}t)=\beta^n h(\alpha^m\beta^{-n}t)$$

for t in the interval $[-1, 1]$ and hence $h(\alpha^m\beta^{-n}t)=\alpha^m\beta^{-n}h(t)$.

If $\log\beta/\log\alpha$ is irrational, we can for s in the interval $(-1, 1)$ choose sequences $\{m_k\}$, $\{n_k\}$ of natural numbers such that $\lim_{k\to\infty}\alpha^{m_k}\beta^{-n_k}=|s|$. It follows that

$$h(s)=\lim_{k\to\infty} h(\alpha^{m_k}\beta^{-n_k})=\lim_{k\to\infty}\alpha^{m_k}\beta^{-n_k}h(1)=sh(1)$$

if $s>0$, and that

$$h(s)=\lim_{k\to\infty} h(\alpha^{m_k}\beta^{-n_k}(-1))=\lim_{k\to\infty}\alpha^{m_k}\beta^{-n_k}h(-1)=sh(-1)$$

if $s<0$ and h is as in (i).

If $\log\beta/\log\alpha=l/k$ where k and l are natural numbers, then $\alpha=\delta^k$ and $\beta=\delta^l$ for some δ in the interval $(0, 1)$. Let d be the greatest common divisor of k and l and write $d=uk+vl$ where u and v are integers. Then $\delta^d=\alpha^u\beta^v$ and as above we see that $h(\delta^d t)=\delta^d h(t)$. Thus, with $\gamma=\delta^d$, the function h is γ-affine and α, β are powers of γ.

Any non-zero t in the interval $(-\gamma, \gamma)$ can be written as $t=\gamma^i s$ where i is a natural number and $|s|$ is in the interval $(\gamma, 1)$. For $t>1$,

$$|h(t)-h(1)\,t|=|h(\gamma^i s)-\gamma^i sh(1)|=\gamma^i\,|h(s)-h(1)\,s|.$$

If h is γ-affine for γ arbitrarily close to 1, it follows that $h(t)=h(1)\,t$ and similarily $h(t)=h(-1)\,t$ for $t<0$ so that h is as in (i). Thus there is a largest number γ for which h is γ-affine.

Suppose h is ρ-affine for a negative number ρ. Since h is also ρ^2-affine we have $\rho^2=\gamma^k$, where k is a natural number. It follows that there is a negative number μ for which h is μ-affine, where $|\mu|$ is as large as possible. If $\rho\neq\mu$, the function h is $\rho\mu^{-1}$-affine and hence there is a natural number l such that $\rho=\gamma^l\mu$.

Statement (iii) is clear. To prove (iv) suppose first that $\lambda>0$. For t in the sets $[-\lambda^n, -\lambda^{n+1}]\cup[\lambda^{n+1}, \lambda^n]$, $n=1, 2, 3, \ldots$ write $t=\lambda^n s$ where s is in the set $[-1, -\lambda]\cup[\lambda, 1]$ and put $h(t)=\lambda^n\psi(s)$. Let also $h(0)=0$.

For $\lambda<0$ we first extend ψ to the set $[-1, -\lambda^2]\cup[\lambda^2, 1]$. For t in the set $[\lambda, -\lambda^2]\cup[\lambda^2, -\lambda]$ write $t=\lambda s$ where s is in the set $[-1, \lambda]\cup[-\lambda, 1]$ and put $\psi(t)=\lambda\psi(s)$. Now extend ψ as above with λ^2 in place of λ.

Finally the statements about differentiability of h are easily seen to be true.

Let us now look at the case of a non-differentiable operating function. If h is an operating function for a subspace B of $C(X)$, where X is compact and B contains the constant functions, the convolution $h_\phi = h * \phi$ of h with a C_0^∞-function ϕ is a smooth operating function for B if the support of ϕ is contained in a sufficiently small neighbourhood of 0. In the locally compact case h_ϕ will no longer be an operating function even though B is uniformly closed. For b in B the function $h_\phi \circ b$ given by

$$(h_\phi \circ b)(x) = \int h(b(x)-s)\, \phi(s)\, ds$$

for every x in X is in general not in B. Instead we look at the functions $h_{\phi, b, c}$ which for b, c in B and a C_0^∞-function ϕ are defined by

$$h_{\phi, b, c}(x) = \int h(c(x)-sb(x))\, \phi(s)\, ds$$

for all x in X. Approximating the integral by Riemann sums we see that if the norm of c is sufficiently small and if the support of ϕ is contained in a sufficiently small neighbourhood of 0 so that $h \circ (c-sb)$ is in B for all s in the support of ϕ, the functions $h_{\phi, b, c}$ are in B. These functions will play a part of the role played by the functions $h_\phi \circ b$ in the compact case.

For a function b_0 in B we let

$$B(b_0) = \{b \in B \mid |b| \leqslant k\, |b_0| \text{ for some } k > 0\}.$$

In general the only functions in $B(b_0)$ might be the functions rb_0, where r is a real number.

If h is an operating function for B, non-affine in every neighbourhood of 0, we find for each open interval I containing 0, a C_0^∞-function ϕ with support contained in I and a real number t in I, such that $(h * \phi'')(t) \neq 0$. If not, h would be twice differentiable with $h'' = 0$ in a neighbourhood of 0. For b_0 in B the function

$$d = \int h \circ (rb_0 - sb_0)\, \phi''(s)\, ds$$

is in B if r is sufficiently small and if the support of ϕ is contained in a sufficiently small neighbourhood of 0 and if $b_0(x) \neq 0$ we can choose r and ϕ such that $d(x) \neq 0$. We note that $d(x) = 0$ if $b_0(x) = 0$. We also note that we can regard the functions in $B(b_0)$ as functions in $C_0(Y)$, where Y is the set of those x in X where $b(x) \neq 0$, and that $C_0(Y)$ can be regarded as a subspace of $C_0(X)$.

As we noted above the space $B(b_0)$ might be trivial. This is not the case if B has a λ-affine operating function. As was noted in Proposition 2 any λ-affine operating function h defined on the interval $[-1, 1]$ satisfies

$$|h(t)| \leqslant k\,|t|$$

for all t in $[-1, 1]$ and some positive number k. In this case $B(b_0)$ contains the functions $h \circ (tb_0)$ for small values of the real number t.

We are going to examine the case where there is an operating function satisfying the growth condition above.

LEMMA 1. *For c_1, c_2 in $B(b_0)$ the function which takes the value $c_1(x)\,c_2(x)\,d(x)/b_0^2(x)$ if $b_0(x) \neq 0$ and the value 0 otherwise is in $cl(B(b_0))$, the uniform closure of $B(b_0)$.*

Proof. Let $c \in B(b_0)$. For sufficiently small real numbers r, s and a C_0^∞-function ϕ with support in a sufficiently small neighbourhood of 0, the function

$$c_t = \int h \circ (rb_0 + tc - sb_0)\,\phi(s)\,ds$$

is in $cl(B(b_0))$. For δ small let

$$\Delta_\delta c_t = (1/\delta^2)(c_{t+\delta} + c_{t-\delta} - 2c_t).$$

If $b_0(x) = 0$ then $\Delta_\delta c_t(x) = 0$. For $b_0(x) \neq 0$, using a change of variables and a mean value theorem for the second derivative, we find that

$$\Delta_\delta c_t(x) = \frac{c^2(x)}{b_0^2(x)} \int h(rb_0(x) + (t+u\delta)\,c(x) - sb_0(x))\,\phi''(s)\,ds$$

where u, depending on x, is in the interval $(-1, 1)$. We put $t = 0$ and deduce that

$$\lim_{\delta \to 0} \Delta_\delta c_0(x) = \frac{c^2(x)}{b_0^2(x)}\,d(x).$$

The function Δc which takes the value $c^2(x)\,d(x)/b_0^2(x)$ if $b_0(x) \neq 0$ and the value 0 otherwise is continuous and is the pointwise bounded limit of the functions $\Delta_\delta c_0$. Hence Δc is in B and since d is in the closure of $B(b_0)$ so is the function Δc. Since $2c_1c_2 = (c_1 + c_2)^2 - c_1^2 - c_2^2$ the result follows.

PROPOSITION 3. *Let B be a uniformly closed subspace of $C_0(X)$ with an operating function h, non-affine in every neighbourhood of 0, with the* *property that $|h(t)| \leqslant k\,|t|$ for all t in a neigbourhood of 0, let b_0 be in B and let Y be the set of those x in X where b_0 does not vanish. Then every f in $C_0(Y)$, which can be approximated by elements from B on an arbitrary set of constancy for the functions c/b_0 where c is in $B(b_0)$, is in B*

Proof. Let A be the set of those bounded continuous functions g on Y with the property that gc is in $cl(B(b_0))$ for every c in $cl(B(b_0))$. By Lemma 1 the functions cd/b_0^2 are in A. Since d/b_0 is in A and since $d/b_0 = b_0 d/b_0^2$ every set of constancy for A is contained in a set of constancy for the functions c/b_0. The result now follows from Machado's theorem.

Let us look the sets of constancy for the functions c/b_0, where c is in $B(b_0)$. Let x_1, x_2 be in a set of constancy for these functions and suppose that $b_0(x_1) \neq b_0(x_2)$. With $c = h \circ (tb_0)$, where t is a real number, we deduce that

$$h(tb_0(x_1))\, b_0(x_2) = h(tb_0(x_2)\, b_0(x_1)$$

for all t in a neighbourhood of 0. We can assume that $b_0(x_1)/b_0(x_2) = \lambda$ where λ is in the half-open interval $[-1, 1)$. We let $s = tb_0(x_2)$ and deduce that

$$h(\lambda s) = \lambda h(s)$$

for all s in a neighbourhood of 0. Thus h is either odd or λ-affine in a neighbourhood of 0.

We are now in a position to settle the case where there is an operating function h satisfying the growth condition $|h(t)| \leqslant k\,|t|$ for all t in a neighbourhood of 0.

THEOREM 5. *Let B be a uniformly closed subspace of $C_0(X)$. If there is an operating function h for B, which is not λ-affine in any neighbourhood of 0 for any $\lambda \in (-1, 1)$, with the property that $|h(t)| \leqslant k\,|t|$ for t in a neighbourhood of 0, every function f in $C_0(X)$, such that $f(x_1) = -f(x_2)$ whenever $b(x_1) = -b(x_2)$ for all b in B, is in B.*

Proof. Let b_0 be in B. For every real number t with $|t|$ sufficiently small, $h \circ (tb_0)$ is in $B(b_0)$. Since h is not λ-affine the conciderations above show that if x_1, x_2 are in the same set of constancy for the functions c/b_0 then, either $b_0(x_1) = b_0(x_2)$ or $b_0(x_1) = -b_0(x_2)$. Thus by Proposition 3, b_0^3 is in B. The result now follows from Theorem 3.

COROLLARY 1. *Let B be a uniformly closed subspace of $C_0(X)$ which separates the points of X. Suppose there is an operating function h for B,* *which is neither odd nor λ-affine in any neighbourhood of* 0 *for any* $\lambda \in (-1, 1)$, *satisfying* $|h(t)| \leqslant k\,|t|$ *for some positive number* k *and all* t *in a neighbourhood of* 0. *Then* $B = C_0(X)$.

Before we look at the case where the operating function is λ-affine we need a result which shows that if there is one λ-affine operating function for a subspace B of $C_0(X)$, there are many.

PROPOSITION 4. *Let* B *be a uniformly closed subspace of* $C_0(X)$ *and let* h *be a non-affine* λ-*affine operating function for* B. *If* k, *defined and continuous in a neighbourhood of* 0, *is* ρ-*affine whenever* h *is* ρ-*affine*, k *is an operating function for* B.

Proof. Let k be as in the proposition and let b_0 be a function in B for which $k \circ b_0$ is defined. Let T be a set of constancy for the functions c/b_0, where c is in $B(b_0)$. For x_1, x_2 in T with $|b_0(x_1)| \leqslant |b_0(x_2)|$, we have

$$h(tb_0(x_1))/b_0(x_1) = h(tb_0(x_2))/b_0(x_2)$$

for all t in a neighbourhood of 0, and hence $h(\rho t) = \rho h(t)$, where $\rho = b_0(x_1)/b_0(x_2)$, for t in a neighbourhood of 0. It follows that $(k \circ b_0)(x_1) = \rho \cdot (k \circ b_0)(x_2)$. We deduce that on T, $k \circ b_0$ is a constant multiple of b_0. By Proposition 3, the function $k \circ b_0$ is in B.

THEOREM 6. *Let* B *be a uniformly closed subspace of* $C_0(X)$. *If there is a* λ-*affine operating function for* B *which is non-affine in every neighbourhood of* 0, *then every function in* $C_0(X)$ *which can be approximated on every pair of points in* X *by a function in* B, *is in* B.

Proof. If h is as in Proposition 2, part (i), replacing $h(t)$ by $h(t) + \alpha t$ or by $-h(t) + \alpha t$, we can assume that $h(t) = \mu\,|t|$ and use Theorem 1.

The case where h is λ-affine for positive values of λ only has a simple proof, which we give first. Since h is λ-affine, $|h(t)| \leqslant k\,|t|$ for some positive number k and all t where h is defined. Also, since h is λ-affine for positive values of λ only, h is not an odd function in any neighbourhood of 0. By Proposition 2 there is a largest number γ in the interval (0, 1) for which h is γ-affine. Let b_0 be in B. For x_1, x_2 in X with $|b_0(x_1)| \leqslant |b_0(x_2)|$, let $\mu = b_0(x_1)/b_0(x_2)$ If μ is negative or not a power of γ, there is a real number t such that

$$h(tb_0(x_1))/b_0(x_1) \neq h(tb_0(x_2))/b_0(x_2),$$

if not h would be odd or μ-affine which is impossible by the assumptions on h and by Proposition 2. We conclude that functions of the form c/b_0, where c is in $B(b_0)$, separate points x_1, x_2 in X that are separated by b_0 and where we do not have $b_0(x_1)/b_0(x_2) = \gamma^k$ for any k. Thus b_0 does not change sign on any set of constancy for the functions c/b_0. Applying Proposition 3 we deduce that $|b_0|$ is in B. Since this holds for any b_0 in B we can apply Theorem 1 and the result follows.

In the general case we may assume that the operating function h is defined on the interval $[-1, 1]$. As observed earlier $|h(t)| \leqslant k\,|t|$ for some k and all t in the interval $[-1, 1]$. By Proposition 2 and Proposition 4 we can assume that h is differentiable on $(-1, 1)$ except possibly at 0 and that h' is bounded, $|h'(t)| \leqslant K$ for all $t \neq 0$ in the interval $(-1, 1)$. Let b_0, c be in B and let

$$a_n = (1/\lambda^n)(h \circ (c + \lambda^n b_0) - h \circ c)$$

for every natural number n. If $b_0(x) = 0$, $a_n(x) = 0$ and if $c(x) = 0$, $a_n(x) = h(b_0(x))$. If both $b_0(x)$ and $c(x)$ are different from 0, using the mean value theorem for derivatives we find that $|a_n(x)| \leqslant K\,|b_0(x)|$ and that $\lim_{n\to\infty} a_n(x) = b_0(x)\,h'(c(x))$. In particular a_n is in $B(b_0)$ for every n.

Let us show that if x_1, x_2 in X, with $b_0(x_1)$ and $b_0(x_2)$ both different from 0, are strongly separated by B, then the points x_1, x_2 are not in the same set of constancy for the functions c/b_0 where c is in $B(b_0)$. Let c_0 in B satisfy $c_0(x_1) = 0$ and $c_0(x_2) = 1$ and let $c = sc_0$, where s is a small real number. The function a_n defined above is in $B(b_0)$ and $a_n(x_1) = h(b_0(x_1))$ for each n while $\lim_{n\to\infty} a_n(x_2) = b_0(x_2)\,h'(s)$. By Proposition 2 and Proposition 4 we can choose h and s such that $h'(s) \neq a_n(x_1)/b_0(x_1)$ and hence $a_n(x_1)/b_0(x_1) \neq a_n(x_2)/b_0(x_2)$ for n large enough. It follows that x_1, x_2 are in the same set of constancy for the functions c/b_0 if and only if $b(x_1) = \rho b(x_2)$ for all b in B and some real number ρ.

It follows from Proposition 3 that if ϕ is a continuous and bounded function on the set Y of those points in X where b_0 does not vanish and if $\phi(x_1) = \phi(x_2)$ whenever x_1, x_2 are in the same set of constancy for the functions c/b_0, the function ϕb_0 is in B and hence also in $B(b_0)$.

Suppose f in $C_0(X)$ can be approximated on every pair of points in X by functions in B. We choose $\{b_n\}$, a sequence of functions in B, such that $f(x) = 0$ if $g(x) = 0$, where $g = \sum (1/2^n)(1/\|b_n\|)\,|b_n|$. Let

$$B(g) = \{b \in B \mid |b| \leqslant k\,|g| \text{ for some } k > 0\}.$$

We are going to show that f is in $B(g)$.

Let

$$Y = \{x \in X \mid g(x) \neq 0\}$$

and let

$$C = \{\phi \in C_b(Y) \mid \phi b \in B \text{ for all } b \in B(g)\}.$$

We want to show that $\phi=|a|/g$ is in C if a is in $B(g)$. Let b_0 be in $B(g)$ and let x_1, x_2 be in Y with $b(x_1)=\rho b(x_2)$ for all b in B and some real number ρ. Then $|a(x_1)|=|\rho|\,|a(x_2)|$ and $g(x_1)=|\rho\,|\,g(x_2)|$ and thus $\phi(x_1)=\phi(x_2)$. As we saw above this implies that ϕb_0 is in B.

Next we want to show that if x_1, x_2 are in the same set of constancy for C, there is a number ρ such that $b(x_1)=\rho b(x_2)$ for all b in B. If there is no such number ρ, then B separates x_1, x_2 strongly so that there is a function a_1 in B with $a_1(x_1)=0$ and $a_1(x_2)=1$. Let $a=sa_1$, where s is a small real number and let

$$a_n=(1/\lambda^n)(h\circ(a+\lambda^n g)-h\circ a)$$

for every natural number n. If $g(x)=0$, $a_n(x)=0$ and if $a(x)=0$, $a_n(x)=h(g(x))$. If $a(x)$ and $g(x)$ are both different from 0, $\lim_{n\to\infty} a_n(x)=g(x)\,h'(x)$. Let $\phi_n=|a_n|/g$, an element of $C_b(Y)$. The calculations above show that $\phi_n(x_1)=|h(g(x_1))|/g(x_1)$ and that $\lim_{n\to\infty}\phi_n(x_2)=|h'(a(x_2))|$. As in the first part it follows from Proposition 2 and Proposition 4 that it is possible to choose h and s so that ϕ_n separates x_1 and x_2 for n large. Also, if y_1, y_2 are in Y with $b(y_1)=\rho b(y_2)$ for all b in B and some real number ρ, we have $a_n(y_1)=\rho a_n(y_2)$ and hence $\phi_n(y_1)=\phi_n(y_2)$. It follows that ϕ_n is in C for each n, a contradiction since x_1, x_2 are in the same set of constancy for C.

Let T be a set of constancy for C, and let x_0 be in T. If x is in T, there is a number ρ such that $b(x_1)=\rho b(x_2)$ for all b in B and hence, since f can be approximated on the set $\{x_0, x\}$ by a function in B, $f(x_1)=\rho f(x_2)$. It follows that if b is in B with $b(x_0)\neq 0$, the function f is a multiple of b on T. By Machado's theorem, f is in B.

Corollary 2. *Let B be a uniformly closed subspace of $C_0(X)$ which separates the points of X strongly. If there is a λ-affine operating function for B which is non-affine in every neighbourhood of* 0, *then $B=C_0(X)$.*

We have been looking at subspaces having operating functions h satisfying $|h(t)|\leqslant k\,|t|$ in a neighbourhood of 0. The results obtained may be summarized as follows: If a uniformly closed subspace B has an operating function satisfying the growth condition above and if B separates the points of X strongly, either the operating function is affine in some neighbourhood of 0, or else $B=C_0(X)$. If there are points in X which are separated weakly by B, the space B may or may not have trivial operating functions. If f.ex. $b(x_1)=\lambda b(x_2)$ and $b(y_1)=\mu b(y_2)$ for all b in B, an operating function h has to satisfy $h(\lambda t)=\lambda h(t)$ and $h(\mu t)=\mu h(t)$ for all t in a neighbourhood of 0. Depending on λ and μ there may or may not exist a function h with this property by Proposition 2. If there is a number λ in the interval $(-1, 1)$ such that $b(x_1) = \lambda^m b(x_2)$ for all b in B (where m may depend on the points x_1, x_2), whenever x_1, x_2 are weakly separated by B and if there is a λ-affine operating function for B which is non-affine in every neighbourhood of 0, every function f in $C_0(X)$ with the property that $f(x_1) = \lambda^m f(x_2)$ if x_1, x_2 are weakly separated by B, is in B if B is uniformly closed.

In the case where there is an operating function h which does not satisfy the condition $|h(t)| \leqslant k\,|t|$ in a neighbourhood of 0, and thus in particular is not λ-affine, the picture is incomplete, we have only partial results.

Let us say that an operating function h has been *normalized* if h is defined on the interval $[-1, 1]$ and satisfies $|h(t)| \leqslant 1$ for $t \in [-1, 1]$.

THEOREM 7. *Let B be a uniformly closed subspace of $C_0(X)$ separating the points of X and suppose there is a normalized operating function h for B, which is not λ-affine in any neighbourhood of* 0, *satisfying $h(t) \leqslant 0$ for $t \leqslant 0$ and $h(t) > Mt$ for all t in an interval* $(0, \delta)$ *for some* $M > 1$. *Then B contains every function f in $C_0(X)$ with the property that $f(x_1) = -f(x_2)$ whenever $b(x_1) = -b(x_2)$ for all b in B.*

To proof this theorem we need a lemma.

LEMMA 2. *Let Y be a closed subset of X. Then there is a positive number L such that if F and G are disjoint compact subsets of Y, for which there is a function b in B, which vanishes outside Y and satisfies $b \leqslant 0$ on F and $b \geqslant 1$ on G, there is such a function b with $\|b\|_\infty \leqslant L$.*

Proof. Replacing h by $h_1(t) = 1/(1+r)\cdot(h(t)+rt)$ we can assume that h satisfies the additional condition $h(t) > \alpha$ for t in the interval $[\delta, 1]$ and some positive α.

Let b be as in the lemma and put $c = \|b\|_\infty^{-1}\, b$. Then c is in B, $c \geqslant \beta$ on G, where $\beta > 0$, $c \leqslant 0$ on F, c vanishes outside Y and $\|c\|_\infty \leqslant 1$. We are going to show that there exists a function with these properties with $\beta = \min\{\alpha, \delta\}$. This will prove the lemma.

If $\beta \geqslant \delta$, c has the right properties. Otherwise, let x be in G. We have $h(c(x)) > \alpha$ if $c(x) \geqslant \delta$ and $h(c(x)) > Mb(x) \geqslant M\beta$ if $c(x) < \delta$. Thus $h \circ c \geqslant \min\{\alpha, M\beta\}$ on G. If $\alpha \leqslant M\beta$, the function $h \circ c$ has the required properties, otherwise we replace c by $h \circ c$ and β by $M\beta$ in this argument. Since $M > 1$ we obtain the desired function in finitely many steps.

Proof of Theorem 7. We are going to show that b^3 is in B for all b in B. The conclusion then follows from Theorem 3.

Let b_0 be in B and put $Y = \{x \in X \mid b_0(x) \neq 0\}$. Let F, G be disjoint compact subsets of Y and let B_1 be the space of those functions in $C_0(X)$ which vanish outside Y. Let further F, G be disjoint compact subsets of Y

and let B_2 be the uniform closure of the space obtained by restricting the functions in B_1 to $F \cup G$. There is a positive number γ such that $|b_0(x)| > \gamma$ for all $x \in F \cup G$, and thus there is for each b in B_1 a positive number k such that $|b| \leqslant k\,|b_0|$ on $F \cup G$. It follows as in Lemma 1 that the restriction to $F \cup G$ of $c_1 c_2 d/b_0^2$ is in B_2 for each pair c_1, c_2 in B_1, where d is as in Lemma 1. As in the proof of Proposition 3, we find that B_2 contains every continuous function on $F \cup G$ which can be approximated on an arbitrary set of constancy for the restrictions to $F \cup G$ of the functions c/b_0, where c is in B_1.

We want to show that these functions separate two points x_1, x_2 in $F \cup G$ unless $b(x_1) = -b(x_2)$ for all b in B. If $b_0(x_1) \neq \pm b_0(x_2)$ we can take $c = b_0 + h \circ (rb_0)$. Since h is not λ-affine a suitable choice of r will give separation. If $b_0(x_1) = \pm b_0(x_2)$ we take $c = h \circ (rc + sb_0) - h \circ (rc)$, where c is a function in B with $c(x_1) = 0$ and $c(x_2) = 1$. Such a function c exists because, since h is not λ-affine, x_1, x_2 are strongly separated by B unless $b(x_1) = -b(x_2)$ for all b in B. If we do not get separation for any choice of r, s in some neighbourhood of 0, the function h must either satisfy $h(r+s) = h(r) + h(s)$ or $h(r-s) = h(r) + h(s)$ for all r, s in some neighbourhood of 0. Either of these conditions imply that h is affine in a neighbourhood of 0, contradicting the assumptions on h.

We conclude that every continuous function ϕ on $F \cup G$ satisfying $\phi(x_1) = -\phi(x_2)$, if x_1, x_2 are in $F \cup G$ and $b(x_1) = -b(x_2)$ for all b in B, is in B_2.

Let f be a function in $C_0(X)$ with $\|f\|_\infty \leqslant 1$, which vanishes outside Y and satisfies $f(x_1) = -f(x_2)$ if $b(x_1) = -b(x_2)$ for all b in B. By the argument above and Lemma 2 there is a function b in B_1, with $\|b\|_\infty \leqslant 2L$, such that $b(x) \leqslant -1$ if $f(x) \leqslant -1/2$ and $b(x) \geqslant 1$ if $f(x) \geqslant 1/2$.

For r between 0 and 1 we have

$$|f - rb| \leqslant 1 - r \qquad \text{on} \quad F \cup G$$

and

$$|f - rb| \leqslant 1/2 + r \cdot 2L \text{ on } X \setminus (F \cup G).$$

We choose r such that $1/2 + r \cdot 2L < 1 - r < 1$ and put $s = 1 - r$. The function $b_1 = rb$ is in B_1, $\|f - b_1\|_\infty \leqslant s$ and $\|b_1\|_\infty \leqslant r \cdot 2L$. By induction we choose a sequence $\{b_n\}$ of functions in B_1, with $\|b_n\|_\infty \leqslant r \cdot 2L$ satisfying

$$\|f - (b_1 + sb_2 + \cdots + s^{n-1} b_n)\|_\infty \leqslant s^n$$

for each n. It follows that f is in B_1. This holds in particular for $f = b_0^3$. This concludes the proof of Theorem 7.

As an example we see that if $b^{1/3}$ is in B for every b in B, then B contains every f in $C_0(X)$ with the property that $f(x_1) = -f(x_2)$ whenever $b(x_1) = -b(x_2)$ for all b in B.

We can obtain the same conclusion as in Theorem 7 with additional conditions on the subspace instead of the operating function.

THEOREM 8. *Let B be a uniformly closed subspace of $C_0(X)$ separating the points of X and having the following property: There is for each x in X a compact neighbourhood K_x of x and functions a_x, b_x in B such that $|b_x| > 0$ on K_x, $a_x = 0$ on $X \setminus K_x$ and $a_x(x) \neq 0$. If there is an operating function for B which is non-affine in every neighbourhood of 0, B contains every f in $C_0(X)$ with the property that $f(x_1) = -f(x_2)$ whenever $b(x_1) = -b(x_2)$ for all b in B.*

Proof. Let y be a point in X and let K_y, a_y, b_y be as in the theorem. For b, c in B, for r_1, r_2, t sufficiently small real numbers and for a C_0^∞-function ϕ with support in a sufficiently small neighbourhood of 0 the function

$$c_t = \int (h \circ (r_1 b + r_2 a_y + tc - sb_y) - h \circ (r_1 b + tc - sb_y))\, \phi(s)\, ds$$

is in B. As in the proof of Lemma 1 we let

$$\Delta_\delta c_t = (1/\delta^2)(c_{t+\delta} + c_{t-\delta} - 2c_t).$$

If $a_y(x) = 0$ we have $\Delta_\delta c_t(x) = 0$, otherwise we find as in the proof of Lemma 1 that

$$\Delta_\delta c_t(x) = \frac{c^2(x)}{b_y^2(x)} \int (h(r_1 b(x) + r_2 a_y(x) + (t + u\delta)\, c(x) - sb_y(x)) - h(r_1 b(x) + (t + u\delta)\, c(x) - sb_y(x)))\, \phi''(s)\, ds,$$

where u, depending on x, is in the interval $(-1, 1)$. Putting $t = 0$ and letting δ tend to 0 we deduce that

$$\lim_{\delta \to 0} \Delta_\delta c_0(x) = \frac{c^2(x)}{b_y(x)}\, d(x),$$

where

$$d(x) = \int (h(r_1 b(x) + r_2 a_y(x) - sb_y(x)) - h(r_1 b(x) - sb_y(x)))\, \phi''(s)\, ds.$$

Since $a_y = 0$ outside the compact set K_y and since $|b_y| > 0$ on K_y, the function Δc, which takes the value 0 on $X \setminus K_y$ and the value $c^2(x)\, d(x)/b_y^2(x)$ for x in K_y, is the pointwise bounded limit of the functions $\Delta_\delta c_0$ and is thus in B. It follows that $c_1 c_2 d/b_y^2$ is in B whenever c_1, c_2 are in B.

Let

$$A = \{f \in C_b(X) \mid fb \in B \text{ for all } b \in B\}.$$

We want to find out which points are separated by A. The algebra A contains the functions cd/b_y^2. For any neighbourhood I of 0 we can choose a C_0^∞-function ϕ with support contained in I and numbers r, s in I, such that $(h_1 * \phi'')(t) \neq 0$ where $h_1(t) = h(t+r) - h(t)$. If not, $h * \phi$ would be constant for all ϕ with support contained in some neighbourhood of 0, and hence h would be constant in a neighbourhood of 0. Thus, given two points x, y in X, we can choose b, r_1, r_2 and ϕ such that $d(y) \neq 0$. Unless $b(x) = -b(y)$ for all b in B we can then choose c such that $c(x) = 0$ and $c(y) = 1$ and thus have a function, cd/b_y^2, in A separating x and y. The conclusion of Theorem 8 now follows from Machado's theorem.

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